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# MEASURES OF IMPORTANCE OF EVENTS AND CUT SETS IN FAULT TREES

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MEASURES OF IMPORTANCE OF EVENTS AND CUT SETS IN FAULT TREES*

H. E. Lambert

Abstract. A survey of the available methods that quantitatively rank basic events and cut sets according to their importance is presented. Such a ranking permits identification of events and cut sets that significantly contribute to the occurrence of the top event. Time-dependent behavior of each method is shown, assuming proportional hazard rates and unrepairable components. A method is presented to compute the importance of events in which repair is permitted. The practical application of importance measures in upgrading system designs and generating checklists for system diagnosis is considered.

1. Introduction. A system is an orderly arrangement of components that performs some task or function. It is clear by the arrangement of these components that some are more critical with respect to the functioning of the system than others. For example, when considering reliability, a component placed in series with the system generally plays a much more important role than that same component placed in parallel with the system. Another factor determining the importance of a component in a system is the reliability of the component, i.e., the probability that the component is working successfully. Measuring the

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relative importance of components may

- Identify components that merit additional research and development, thereby improving the overall reliability at minimum cost or effort
- Suggest the most efficient way to diagnose system failure by generating a repair checklist for an operator to follow.

The fault tree is the most generalized Boolean model capable of identifying those basic causes that can contribute to system failure. These basic causes or events include environmental conditions, human error, and normal events (events that are expected to occur during the life of the system) as well as hardware failures. If the relative failure rates of the basic events are known, the fault tree can be quantitatively evaluated to assess their importance (see Lambert [1] for a discussion of fault tree analysis).

Several probabilistic methods can be used to compute the importance of basic events in the fault tree. All the methods assess the importance of basic events by a numerical ranking. The probabilistic interpretation describing the relationship of the occurrence of a basic event to the occurrence of the top event is different in each case.

2. Mathematical notation.* Each probabilistic method that computes importance can be expressed in terms of a "g" function that computes the probability of the top event in terms of the basic event probabilities. To generate this function we need a Boolean expression for the top

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*The notation in this section is that of coherent structure theory, see Ref. [2].

event in terms of the Boolean variables of the basic events. The outcome of each basic event at time t has an indicator variable $Y_i(t)$,

$$Y_i(t) = \begin{cases} 1 \text{ when basic event i is occurring at time t} \\ 0 \text{ otherwise} \end{cases}$$

and the top event has an indicator variable $\psi(\underline{Y}(t))$,

$$\psi(\underline{Y}(t)) = \begin{cases} 1 \text{ when the top event is occurring at time t} \\ 0 \text{ otherwise} \end{cases}$$

where $\underline{Y}(t) = (Y_1(t), Y_2(t), \ldots, Y_n(t))$ is the vector of basic event outcomes at time t and n is the number of basic events in the fault tree.

There are two basic operators that express $\psi(\underline{Y}(t))$ in terms of $\underline{Y}(t)$: the AND operator, $\prod$, and the OR operator, $\coprod$. Figure 1 shows an example of the $\prod$ operator.

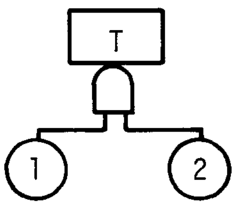


Fig. 1. AND gate with two inputs.

In this case the top event, T, occurs if basic events 1 and 2 occur. The structure function is given by

$$\psi(\underline{Y}(t)) = \psi(Y_1(t), Y_2(t)) = \prod_{i=1}^{2} Y_i(t) = Y_1(t) \cdot Y_2(t) .$$

In general, the structure function for an AND gate with n inputs (Fig. 2) is given by

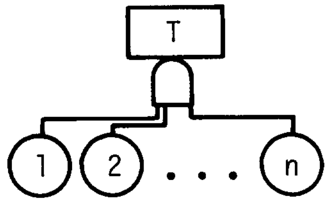


Fig. 2. AND gate with n inputs.

$$\psi(\underline{Y}(t)) = \prod_{i=1}^{n} Y_i(t) = Y_1(t)\cdot Y_2(t)\cdot \ldots \cdot Y_n(t) .$$

Figure 3 shows an example of the $\amalg$ operator. In this case, the top event occurs if basic event 1 or 2 occurs. The structure function is given by

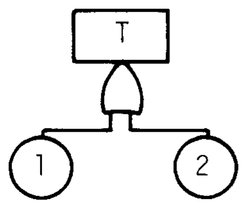


Fig. 3. OR gate with two inputs.

$$\psi(\underline{Y}(t)) = \coprod_{i=1}^{2} Y_i(t) \overset{\text{def}}{\equiv} 1 - \prod_{i=1}^{2} (1 - Y_i(t)) = Y_1(t) + Y_2(t) - Y_1(t)\cdot Y_2(t) .$$

In general, the structure function for an OR gate with n inputs (Fig. 4) is given by

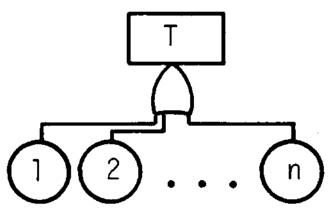


Fig. 4. OR gate with n inputs.

$$\psi(\underline{Y}(t)) = \coprod_{i=1}^{n} Y_i(t) \overset{\text{def}}{\equiv} 1 - \prod_{i=1}^{n} (1 - Y_i(t)) .$$

In this section we limit our discussion to basic events that have infinite fault duration times, i.e., when a basic event occurs, it occurs for the life of the system. If the state of each basic event is random, then the probability that event i occurs by time t can be defined to be $F_i(t)$. If $\lambda_i(t)dt$ is defined to be the probability that event i occurs between t and t + dt, given that event i has not occurred by time t, then $F_i(t)$ can be expressed in terms of $\lambda_i(t)$:*

$$F_i(t) = 1 - e^{-\int_0^t \lambda_i(t')dt'} .$$

We recognize from the above definition that $E[Y_i(t)] = \text{Prob}[Y_i(t) = 1] = F_i(t)$, where E is the expected value.

If basic events are statistically independent, then the probability that the top event occurs by time t can be defined as $g(\underline{F}(t))$. Likewise it can be recognized that $E[\psi(\underline{Y}(t))] = \text{Prob}[\psi(\underline{Y}(t)) = 1] = g(\underline{F}(t))$. †

In the case of the AND gate with two inputs, $g(\underline{F}(t))$ is given by

$$g(\underline{F}(t)) = F_1(t) \cdot F_2(t)$$

and for an OR gate with two inputs, $g(\underline{F}(t))$ is given by

$$g(\underline{F}(t)) = 1 - (1 - F_1(t)) \cdot (1 - F_2(t)) .$$

## 3. Probabilistic expressions that measure importance.

### 3.1. Assumptions in quantitative calculations.

In this paper it is assumed that all basic events are statistically independent. Probabilistic evaluations in which basic events are positively dependent (the

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* $\lambda_i(t)$ is commonly referred to as the hazard or failure rate at time t.

† The g function is analogous to the h function in reliability [2].

technical term is association) are discussed by Barlow and Proschan [2]. No generality in methodology is lost, however, if we assume that basic events are statistically independent. Further, it is assumed (unless otherwise indicated) that all basic events have an infinite fault duration time (i.e., in the case of components repair is not permitted). Hence, g is only a function of $\underline{F}(t)$. It is shown later that the same methods apply in describing the importance of events with finite fault duration times.

3.2. Birnbaum's measure of importance. In 1969, Birnbaum introduced the concept of importance for coherent systems [3]. He defined the reliability importance of a component i as the rate at which system reliability improves as the reliability of component i improves. If we construct a fault tree where the top event is system failure and the basic events are component failures,* then Birnbaum's definition of component importance becomes

$$(1) \qquad \frac{\partial g(\underline{F}(t))}{\partial F_i(t)} = g(1_i, \underline{F}(t)) - g(0_i, \underline{F}(t)) .$$

Stated in other terms, the above expression is the probability that the system is in a state in which the functioning of component i is critical: the system functions when i functions, the system fails when i fails. The failure of i is critical when $\psi(1_i, \underline{Y}) - \psi(0_i, \underline{Y}) = 1$.

*At this point it is convenient to denote basic events as component failures when describing methods that measure importance. Used in this context, event importance is synonymous with component importance.

Of interest might be the total number of vector states in which a component is critical. If we fix the state of a component in the system, we are left with $2^{n-1}$ states, where n equals the number of components. In the above expression, if we let $F_j(t) = 1/2$ for all $j \neq i$, then the number of states in which component i is critical, denoted by $B_i$, is

(2) $$B_i = 2^{n-1} \{g(1_i, \underline{1/2}) - g(0_i, \underline{1/2})\} .$$

Birnbaum calls $B_i$ the structural importance of component i.

For example, the fault tree shown in Fig. 5 exhibits three states in which the failure of 1 is critical:

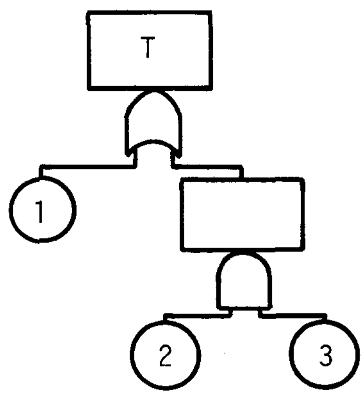


Fig. 5. Fault tree with AND and OR gates.

(1) $Y_2 = 0$ and $Y_3 = 0$

(2) $Y_2 = 1$ and $Y_3 = 0$

(3) $Y_2 = 0$ and $Y_3 = 1$

The number of critical cut sets for component 1 can be determined by using Eq. 2. The structure function $\psi(\underline{Y})$ is given by

$$\psi(\underline{Y}) = Y_1 \amalg Y_2 \cdot Y_3{}^{*}$$

$$= 1 - (1 - Y_1)(1 - Y_2 \cdot Y_3)^{\dagger}$$

for $i = 1$, $B_1 = 2^{3-1} \cdot \{1 - (1 - 3/4)\} = 3$ as verified above.

The sets of events $\{1\}$, $\{1, 2\}$, and $\{1, 3\}$ are known as critical cut sets for component 1. The set $\{2, 3\}$ is a critical cut set for components 2 and 3. Note that a minimal cut set containing i is always a critical cut set for i.**

3.2.1. Criticality importance. Birnbaum's definition of importance is a conditional probability in the sense that the state of the $i^{th}$ component is fixed. The probability that the system is in a state at time t in which component i is critical and that component i has failed by time t is

$$\{g(1_i, \underline{F}(t)) - g(0_i, \underline{F}(t))\}\, F_i(t) .$$

If we make this conditional to system failure by time t, then the above expression becomes

$$\frac{\{g(1_i, \underline{F}(t)) - g(0_i, \underline{F}(t))\}\, F_i(t)}{g(\underline{F}(t))} . \tag{3}$$

The above expression is defined as the criticality importance of component i.

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*$\prod$ and $\coprod$ operate on sets of indicator variables while $\pi$ and $\amalg$ operate on pairs of indicator variables.

†Known as the min cut representation of $\psi(\underline{Y})$. In this case, there are two min cut sets: $\{1\}$ and $\{2, 3\}$.

**For a set of events to be a critical cut set for event i, each cut set contained in this set must contain the event i.

3.3. Vesely-Fussell definition of importance. It is possible that when system failure is observed, two or more cut sets could have failed. In this case, restoring a failed component to a working state does not necessarily mean that the system is restored to a working state. In other words, it is possible that a failure of a component can be contributing to system failure without being critical. Component i is contributing to system failure if a cut set containing i has failed; in terms of coherent structure theory notation

$$\psi_K^i(\underline{Y}(t)) = \coprod_{j=1}^{N_K^i} \prod_{\substack{\ell \epsilon K_j \\ i \epsilon K_j}} Y_\ell(t) = 1 ,$$

where $\substack{\ell \epsilon K_j \\ i \epsilon K_j}$ means that the index $\ell$ includes all basic events in cut set $K_j$, where $K_j$ contains the basic event i.

$N_K^i$ = number of cut sets that contain basic event i

$\psi_K^i(\underline{Y}(t))$ = Boolean indicator variable for the union of all cut sets that contain basic event i.

The probability that component i is contributing to system failure, Prob $[\psi_K^i(\underline{Y}(t)) = 1]$, is denoted as $g_i(\underline{F}(t))$. The probability that component i is contributing to system failure, given that the system has failed by time t, is given by

$$\frac{g_i(\underline{F}(t))}{g(\underline{F}(t))} . \tag{4}$$

This concept of importance was introduced by Vesely [4] and also Fussell [5], who later described it.

Note that if we substitute $g_i(\underline{F}(t))$ for $g(\underline{F}(t))$ in the definition of criticality importance, we obtain

$$\frac{\{g_i(1_i, \underline{F}(t)) - g_i(0_i, \underline{F}(t))\} F_i(t)}{g(\underline{F}(t))} .$$

Noting that

$$g_i(0_i, \underline{F}(t)) = 0$$

$$g_i(1_i, \underline{F}(t))F_i(t) = g_i(\underline{F}(t)) ,$$

we obtain the Vesely-Fussell definition of component importance

$$\frac{g_i(\underline{F}(t))}{g(\underline{F}(t))} .$$

Indeed when component i is contributing to system failure, it is always critical to the structure $\psi_K{}^i(\underline{Y}(t))$.

3.4. Barlow-Proschan measure of importance. Barlow and Proschan [6] take a time series approach in describing component importance. They assume that if components fail sequentially in time and that if two or more components have a vanishingly small probability of occurring at the same instant, then one component must have caused the system to fail. The probability that event i causes the system to fail during a differential time interval of t', where $t' \leq t$, is

$$\{g(1_i, \underline{F}(t')) - g(0_i, \underline{F}(t'))\} dF_i(t') .$$

Integrating between 0 and t

$$\int_0^t \{g(1_i, \underline{F}(t')) - g(0_i, \underline{F}(t'))\} dF_i(t')$$

we get the probability that component i causes the system to fail.

It can be shown that [6]

$$\sum_{i=1}^{n} \int_{o}^{t} \{g(1_i, \underline{F}(t')) - g(0_i, \underline{F}(t'))\} \, dF_i(t')$$

is the probability that the system fails before t, where n is the number of components comprising the system. The conditional probability that a component i causes the system to fail by the time t is then the Barlow-Proschan (B-P) measure of importance

$$(5) \qquad \frac{\int_{o}^{t} \{g(1_i, \underline{F}(t')) - g(0_i, \underline{F}(t'))\} \, dF_i(t')}{\sum_{i=1}^{n} \int_{o}^{t} \{g(1_i \underline{F}(t')) - g(0_i, \underline{F}(t'))\} \, dF_i(t')} .$$

The sum of all component importances in Barlow's measure of importance is unity. Essentially, B-P's measure of importance of a component i is the probability of the system failing because a critical cut set containing i fails, with component i failing last.

It might be interesting to assess the role of the failure of a component i when another component, say j, causes the system to fail. The failure of i is a factor in this case only if i and j are contained in at least one cut set. The probability that component i is critical when j causes the system to fail is

$$\frac{\int_{o}^{t} \{g(1_i, 1_j, \underline{F}(t')) - g(1_i, 0_j, \underline{F}(t'))\} \, F_i(t') \, dF_j(t')}{g(\underline{F}(t))}$$

and, in general, the probability that component i is critical when another component causes the system to fail is

$$(6) \qquad \frac{\sum_{\substack{j \\ i \neq j}} \int_0^t \{g(1_i, 1_j, \underline{F}(t')) - g(1_i, 0_j, \underline{F}(t'))\} F_i(t')\, dF_j(t')}{g(\underline{F}(t))},$$

where the sum over j is to include only those components that appear in at least one cut set with component i.

Barlow and Proschan define the structural importance of component i as the probability that component i causes the system to fail, assuming that all component failure probabilities are equal. They then integrate from time $t = 0$ to $t = \infty$, or equivalently from $q = 0$ to $q = 1$

$$(7) \qquad \int_0^\infty [g(1_i, \underline{q}) - g(0_i, \underline{q})]\, dq,$$

where $q = F(t)$.

4.0. Assumption of proportional hazards. To compare the time-dependent behavior of each method that measures importance, we must know the basic event probabilities, $F_i(t)$, which implies knowledge of $\lambda_i(t)$. In many cases, the failure rates are known to a poor degree of accuracy. However, by predicating the relative failure rates of the basic events on experience, engineering judgement, and existing failure rate data they may become better known. Furthermore, if we assume that all the failure rates exhibit the same time-dependent behavior (assumption of proportional hazards) then $F_i(t)$ may be written as

$$F_i(t) = 1 - e^{-R(t)\lambda_i}$$

for i = 1, 2, ..., n; where R(t) is the common hazard and

$$\lambda_i = \frac{\int_0^t \lambda_i(t')\, dt'}{R(t)} .$$

If we arbitrarily select a reference $\lambda_j$ from $\underline{\lambda}$,* we may express $F_i(t)$ in terms of $F_j(t)$:

$$F_i(t) = 1 - (1 - F_j(t))^{\lambda_i/\lambda_j} .$$

Letting $\alpha_i = \lambda_i/\lambda_j$ and $q(t) = F_j(t)$, $F_i(t)$ becomes

(8) $$F_i(t) = 1 - (1 - q(t))^{\alpha_i} ,$$

where $\alpha_i$ is defined as the proportional hazard for basic event i.

4.1. Time-dependent behavior of importance measures. Under the assumption of proportional hazards, each method can either be plotted as a function of q(t) and $\underline{\alpha}$ or as a function of $g(\underline{F}(t))$ and $\underline{\alpha}$ since $g(\underline{F}(t))$ is a function of q(t) (and $\underline{\alpha}$). We chose three systems to compare each measure of importance. System A is a parallel system with components 1 and 2. The fault tree is shown in Fig. 6a and a reliability network diagram is shown in Fig. 6b.**

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*Where $\underline{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ and n is the number of basic events in the fault tree.

**The fault tree shown in Fig. 6a is failure-oriented; that is, T represents system failure and the numbers in the circle represent component failures. The reliability network diagram in Fig. 6b is success-oriented.

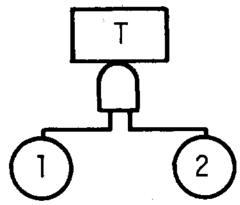


Fig. 6a. System A fault tree; the structure function is $\psi(Y_1, Y_2) = Y_1 \cdot Y_2$.

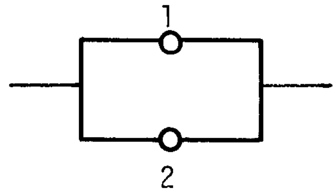


Fig. 6b. Reliability network diagram.

We assume a proportional hazard rate of 0.01 for component 1 and 1 for component 2; i.e., $\alpha_1 = 0.01$, and $\alpha_2 = 1$. In this case, $F_1(t) = 1 - (1 - q(t))^{0.01}$, $F_2(t) = q(t)$, and $g(\underline{F}(t)) = q(t) - q(t)(1 - q(t))^{1.01}$. Five measures of importance are plotted vs $g(\underline{F}(t))$ in Fig. 7. They include Birnbaum, expression (1); criticality, expression (3); Vesely-Fussell, expression (4); Barlow-Proschan, expression (5) and the upgrading function, $\frac{\alpha_i}{g(q(t), \underline{\alpha})} \cdot \frac{\partial g(q(t), \underline{\alpha})}{\partial \alpha_i}$.

The significance of the upgrading function is discussed in the next section when upgrading systems are considered.

We note in Fig. 7 that the probability that each component either contributes to or is critical to system failure is unity in each case. Barlow's and Birnbaum's definition of importance indicates that component 1 is more important. In a parallel system, the system fails when the last component fails; in this case, component 1 is more likely to fail last and cause the system to fail. Birnbaum's measure of importance tells us that system A is most likely to be in a state in which the failure of component 1 is critical.

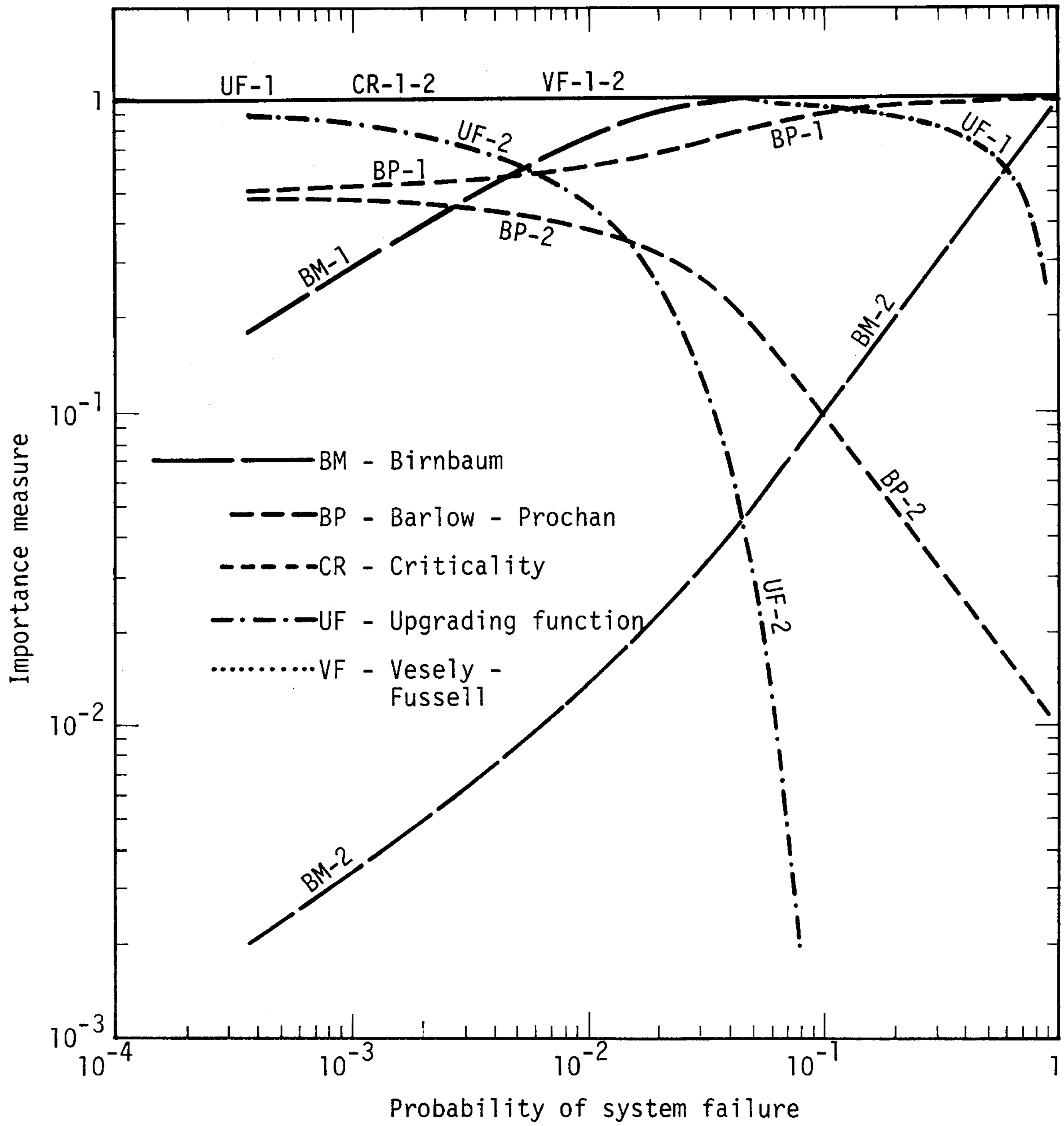


Fig. 7. Plots of importance measures for system A.

System B is a series system of two components, 1 and 2. We can assume the same proportional hazard rate as in system A. In this case $g(\underline{F}(t)) = 1 - (1 - q(t))^{1.01}$. The fault tree and network diagram are shown in Fig. 8.

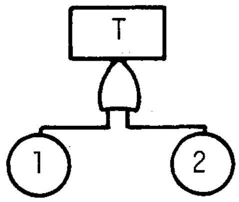


Fig. 8a. System B fault tree; the structure function is $\psi(Y_1, Y_2) = 1 - (1 - Y_1)\cdot(1 - Y_2)$.

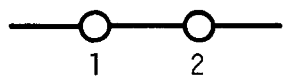


Fig. 8b. Reliability network diagram.

The plots in Fig. 9 show that component 2 is more important than component 1 in all cases. This is to be expected since component 2 has a failure rate 100 times greater than component 1 and a series system fails when the first component fails.

System C is a series-parallel system. Component 1 is in series with a parallel structure of two components, 2 and 3. The fault tree and network diagram are shown in Fig. 10. For this example, it is assumed that $\alpha_1 = 0.1$ and $\alpha_2 = \alpha_3 = 1$. Figure 11 indicates that for small $g(\underline{F}(t))$ or small times t, component 1 is more important.* For large $g(\underline{F}(t))$ $(\gtrsim 0.05)$ or large t, components 2 and 3 are more important. There is disagreement, however, as to which value of $g(\underline{F}(t))$ would make components 2 and 3 more important than component 1.

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*The value t can be thought of as mission time.

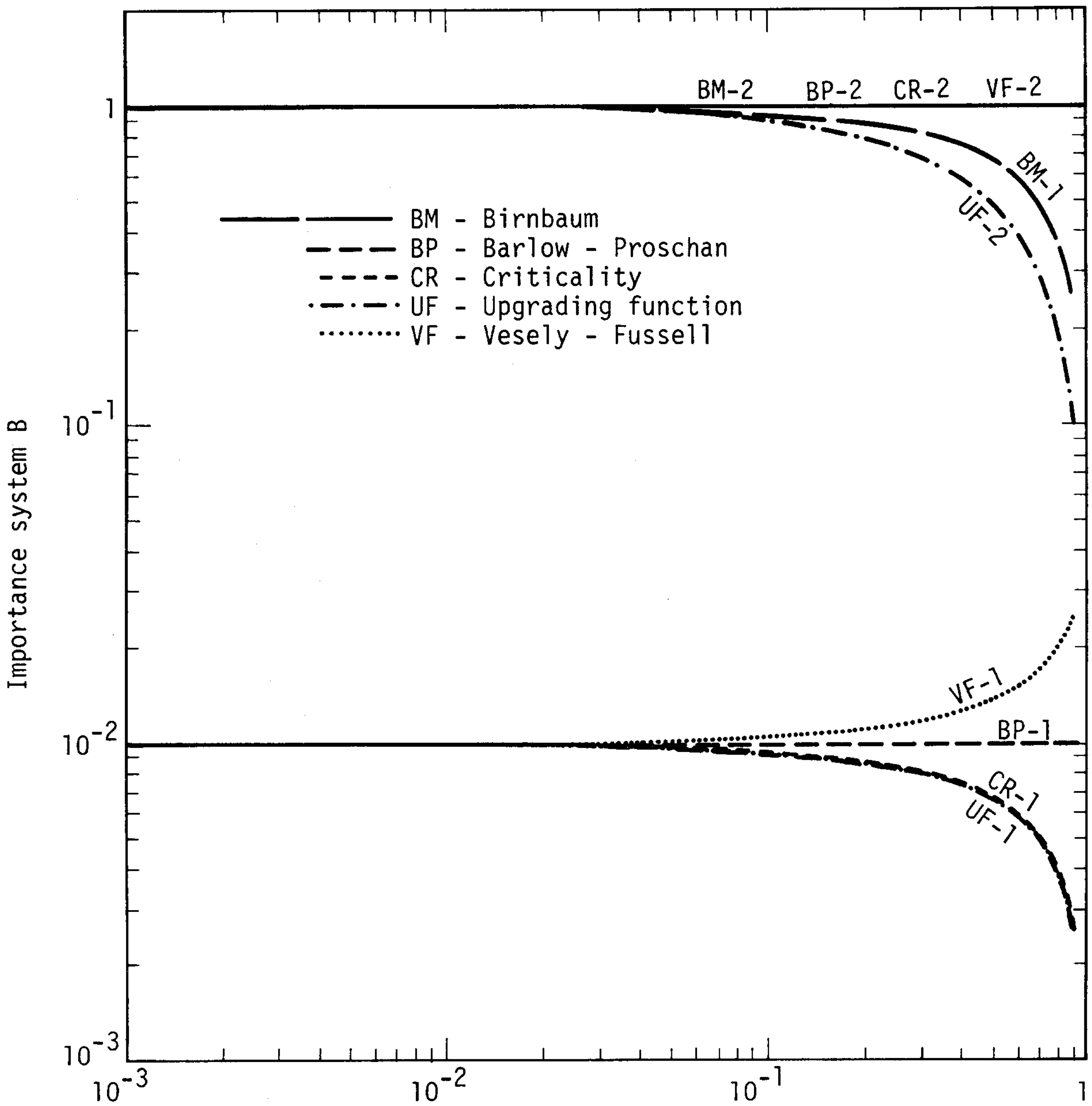


Fig. 9. Plots of importance measures for system B.

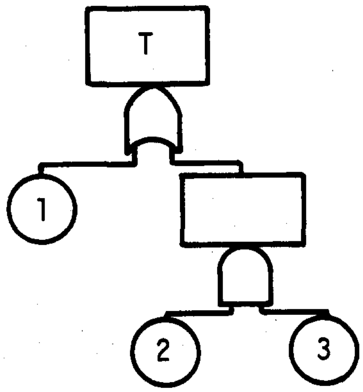


Fig. 10a. System C fault tree; the structure function is $\psi(\underline{Y}) = 1 - (1 - Y_1) \cdot (1 - Y_2 \cdot Y_3)$.

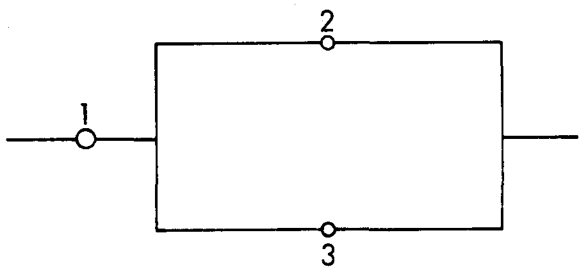


Fig. 10b. Reliability network diagram.

It can be seen from Figs. 7, 9, and 11 that each method has a different time-dependent behavior; i.e., there is disagreement in the assessment of importances. The analyst should carefully define the probabilistic information he seeks regarding his system and then apply the appropriate measure of importance.

4.2. Cut set importance. Definitions of cut set importance are described in terms of methods that determine component importance.

In the Vesely-Fussell definition, the importance of a cut set $K_j$ is the probability that cut set $K_j$ is contributing to system failure. It is given by

(9) $$\frac{\prod_{i \in K_j} F_i(t)}{g(\underline{F}(t))} .$$

The Barlow-Proschan definition of the importance of a cut set $K_j$ is the probability that a cut set $K_j$ causes the system to fail. For a cut set $K_j$ to have caused the system to fail, a basic event contained in the cut set

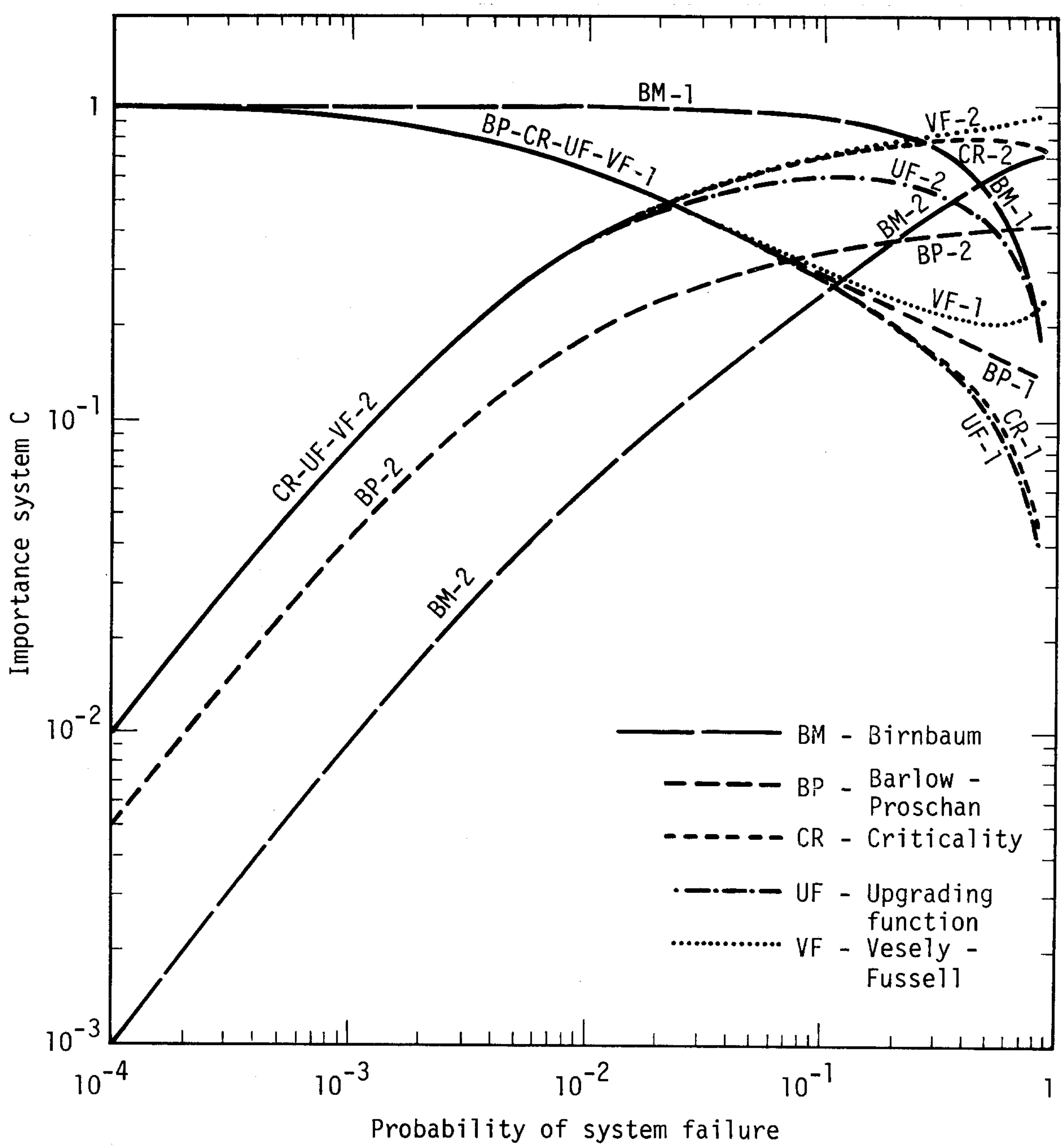


Fig. 11. Plots of importance measures for system C.

must have caused the system to fail and all other events in the cut set must have failed prior to the event that caused the system to fail. B-P's measure of importance of a cut set $K_j$ is

$$\frac{\sum_{i \in K_j} \int_0^t [g(\underline{1}^{K_j}, \underline{F}(t)) - g(0_i, \underline{1}^{K_j - \{i\}}, \underline{F}(t))] \prod_{\substack{j \neq i \\ j \in K}} F_j(t)\, dF_i(t)}{g(\underline{F}(t))},$$

where $\underline{1}^{K_j}$ means that $Y_i$ is equal to 1 for each basic event i contained in cut set $K_j$. Since $g(\underline{1}^{K_j}, \underline{F}(t)) = 1$, the above expression becomes

$$\frac{\sum_{i \in K_j} \int_0^t [1 - g(0_i, \underline{1}^{K_j - \{i\}}, \underline{F}(t))] \prod_{\substack{j \neq i \\ j \in K}} F_j(t)\, dF_i(t)}{g(\underline{F}(t))}. \tag{10}$$

Vesely-Fussell's definition of cut set importance always assigns more importance to a cut set of a lower order than a cut set of a higher order when basic event probabilities are equal.* This is not always true, however, with B-P's measure of importance. As an example, consider a 10 component system with min cut sets given by

| | | | |
|---|---|---|---|
| $K_1 = \{1, 2, 3, 4\}$ | $K_6 = \{5, 7, 8\}$ | $K_{11} = \{5, 9, 10\}$ | $K_{16} = \{6, 8, 10\}$ |
| $K_2 = \{5, 6, 7\}$ | $K_7 = \{5, 7, 9\}$ | $K_{12} = \{6, 7, 8\}$ | $K_{17} = \{6, 9, 10\}$ |
| $K_3 = \{5, 6, 8\}$ | $K_8 = \{5, 7, 10\}$ | $K_{13} = \{6, 7, 9\}$ | $K_{18} = \{7, 8, 9\}$ |
| $K_4 = \{5, 6, 9\}$ | $K_9 = \{5, 8, 9\}$ | $K_{14} = \{6, 7, 10\}$ | $K_{19} = \{7, 8, 10\}$ |
| $K_5 = \{5, 6, 10\}$ | $K_{10} = \{5, 8, 10\}$ | $K_{15} = \{6, 8, 9\}$ | $K_{20} = \{7, 9, 10\}$ |
| | | | $K_{21} = \{8, 9, 10\}$ |

*Order indicates the number of basic events contained in a cut set.

No components of $K_1$ appear in other min cut sets. The remaining sets were obtained by taking all combinations of three components from the remaining six. For this system

$$g(\underline{F}(t)) = \text{Prob}\,[\coprod_{i=1}^{21} \kappa_i = 1] = \text{Prob}\,[\kappa_1 \amalg \coprod_{i=2}^{21} \kappa_i = 1)]$$

$$= 1 - (1 - \text{Prob}\,(\kappa_1 = 1))(1 - \text{Prob}\,(\coprod_{i=2}^{21} \kappa_i = 1))\,,$$

where $\kappa_i$ is the indicator variable for cut set $K_i$. Setting $q(t) = F_i(t)$ for all i, where i = 1 to 10

$$g(\underline{F}(t)) = 1 - (1 - q(t)^4)(1 - \sum_{j=3}^{6} \binom{6}{j} (1 - q(t))^{6-j}\, q(t)^j)\,.$$

Substituting in expression (10), Barlow's measure of importance for cut set $K_1$, $I_{K_1}$ becomes

$$I_{K_1} = \frac{4 \int_0^{q(t)} [1 - \sum_{j=3}^{6} \binom{6}{j} (1 - q')^j\, q'^{6-j})]\, q'^3\, dq'}{g(\underline{F}(t))}\,,$$

for cut set $K_2$

$$I_{K_2} = \frac{3 \int_0^{q(t)} (1 - q'^4)\,(1 - q')^3\, q'^2 dq'}{g(\underline{F}(t))}\,.$$

The Vesely-Fussell definition of importance gives

$$I_{K_1} = \frac{q(t)^4}{g(\underline{F}(t))}\,, \qquad I_{K_2} = \frac{q(t)^3}{g(\underline{F}(t))}\,.$$

In Fig. 12, the importances of cut sets $K_1$ and $K_2$ are plotted as a function of $g(\underline{F}(t))$. Cut set $K_2$ always has a greater probability of <u>contributing</u> to system failure than cut set $K_1$. However, for $g(\underline{F}(t)) \gtrsim 0.64$, cut set $K_1$ has a greater probability of <u>causing</u> the system to fail than cut set $K_2$. Replication of the basic events in cut sets decreases the probability that a cut set can cause the system to fail. If the basic events contained in $K_2$ were not replicated in other cut sets, then $K_2$ would always have a higher failure probability of causing the system to fail than $K_1$. In general, when no replication of events occurs, a lower order cut set is always more important than a higher order cut set when basic event probabilities are equal.

<u>4.3. Importance of components when repair is permitted.</u> Each of the methods previously described can also assess the importance of components when repair is permitted. In every importance expression except Barlow and Proschan's, the limiting unavailability, $\overline{A}_i$, can be substituted for $F_i(t)$ without any change in probabilistic meaning.

To motivate B-P's definition of component importance when repair is permitted, consider an unrepairable system that has failed at some specified time t. If component i has distribution $F_i$ with density $f_i$ (i = 1, 2, ..., n), then the probability that i caused system failure (given that the system failed precisely at time t) is

$$(11) \qquad \frac{[g(1_i, \underline{F}(t)) - g(0_i, \underline{F}(t))]\ f_i(t)\ dt}{\sum_{j=1}^{n} [g(1_j, \underline{F}(t)) - g(0_j, \underline{F}(t))]\ f_j(t)\ dt}.$$

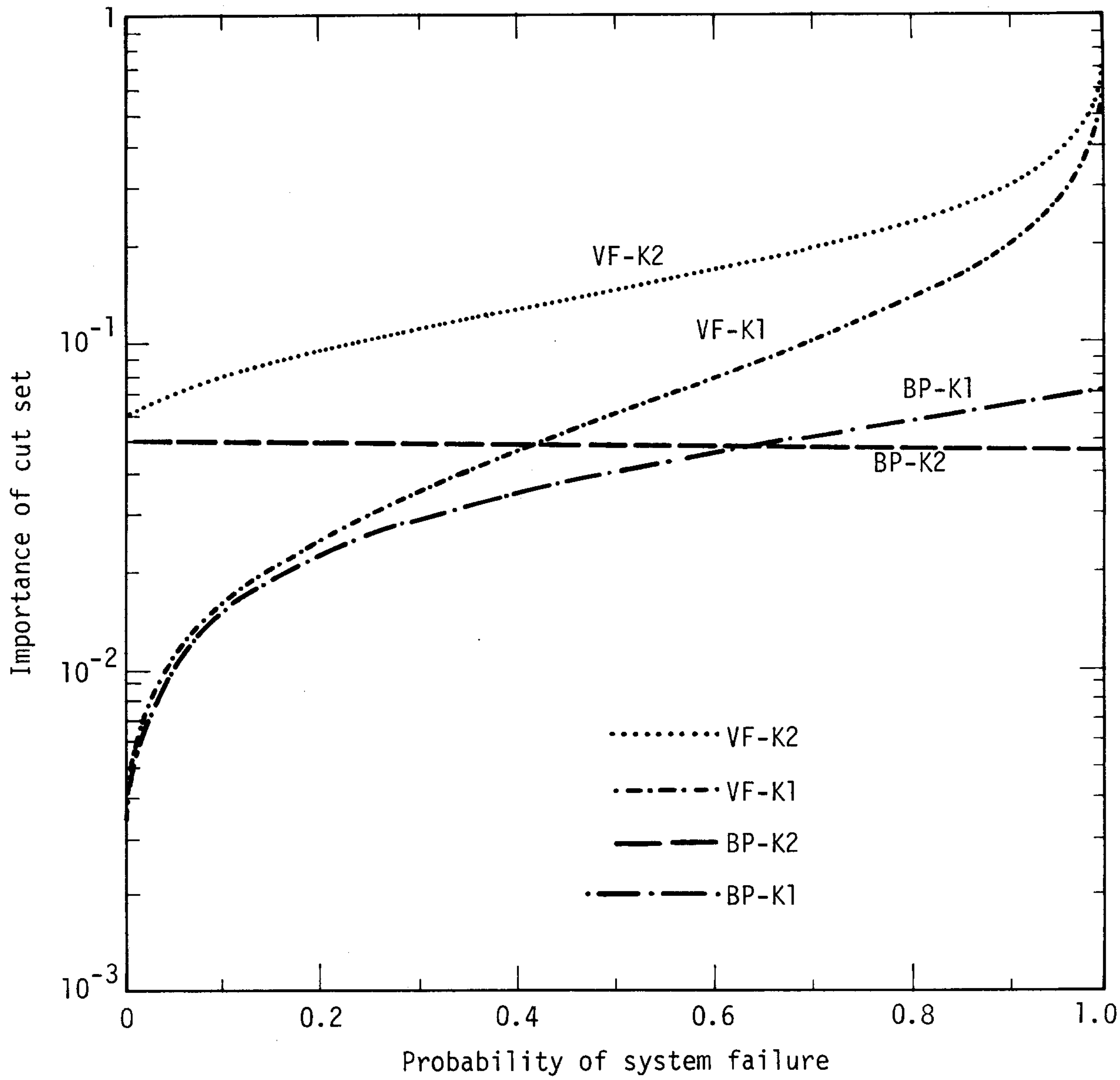


Fig. 12. Plots of cut set importance.

In renewal theory, the process of repairing a failed component is called an alternating renewal process. In this case, the component alternates between two states, an upstate and a downstate. The probability that a renewal occurs about some differential time interval is $dM_i(t)$, called the renewal density.* $dM_i(t)$ is analogous to $f_i(t)$ in the nonrepairable case. The probability that a component is down at time t is $\overline{A}_i(t)$,† called the unavailability of component i at time t, (analogous to $F_i(t)$). The probability that component i caused system failure is

$$(12) \qquad \frac{[g(1_i, \underline{\overline{A}}(t)) - g(0_i, \underline{\overline{A}}(t))]\, dM_i(t)}{\sum_{i=1}^{n} [g(1_i, \underline{\overline{A}}(t)) - g(0_i, \underline{\overline{A}}(t))]\, dM_i(t)} ,$$

where

$\mu_i$ = mean time to failure for component i

$\nu_i$ = mean time to repair for component i

$$\lim_{t \to \infty} \overline{A}_i(t) = \frac{\nu_i}{\nu_i + \mu_i} .$$

Letting $t \to \infty$, we obtain the stationary probability that component i causes system failure [6]

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*$M_i(t) = E[N_i(t)]$, where $N_i(t)$ is the number of failures of component i in [0, t].

†$A_i(t) = E[X_i(t)]$, where $X_i(t) = \begin{cases} 1 & \text{if component } i \text{ is working} \\ 0 & \text{otherwise} \end{cases}$

and $\overline{A}_i(t) = 1 - A_i(t)$

(13) $$\frac{[g(1_i, \underline{A}) - g(0_i, \underline{A})]/(\mu_i + \nu_i)}{\sum_{j=1}^{n}[g(1_j, \underline{A}) - g(0_j, \underline{A})]/(\mu_j + \nu_j)} .$$

As the following discussion shows, theresult is reasonable on physical grounds. $\mu_i + \nu_i$ is the average amount of time between failures for component i; i.e., the average length of time for a renewal cycle. $1/(\mu_i + \nu_i)$ is the average rate at which the renewal process takes place for component i. At large times the system failure probability is time-invariant since the probability that each component fails is time-invariant.

5. Applications of importance measures. The assumption of proportional hazards allows us to make quantitative evaluations from relative rather than absolute information. If the importance measures are not a sensitive function of either $g(\underline{F}(t))$ or $q(t)$ (or if $q(t)$ is known within an order of magnitude), then the importance of each event can be assessed with just the knowledge of proportional hazards.*

5.1. Upgrading systems. An importance calculation can be used to quantitatively determine the basic events that either contribute to, or cause, the occurrence of the top event. Weaknesses inherent in the system can be identified and the system can be optimally upgraded by reducing the importance of these basic events. The importance of an event can be reduced in any one of a number of ways [7].

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*Recall that $q(t) = 1 - e^{-R(t)\lambda_j}$, where $R(t)$ is the common hazard and $\lambda_j$ is the reference failure rate.

Unfortunately, Birnbaum's measure of importance,

$$\frac{\partial g(\underline{F}(t))}{\partial F_i(t)},$$

cannot be practically applied to upgrading reliable systems. For a given incremental reduction $\Delta x$ in $F_i(t)$, Birnbaum identifies the event i that has the greatest effect in reducing $g(\underline{F}(t))$; i.e.,

$$\frac{\partial g(\underline{F}(t))}{\partial F_i(t)}$$

identifies the event i in which the quantity

$$g[F_i(t), \underline{F}(t)] - g[F_i(t) - \Delta x, \underline{F}(t)]$$

is a maximum. Note that the above difference does not depend upon $F_i(t)$ because

$$\frac{\partial g(\underline{F}(t))}{\partial F_i(t)}$$

is not a function of $F_i(t)$ if basic events are statistically independent. Recall that $\frac{\partial g(\underline{F}(t))}{\partial F_i(t)} = g(1_i, \underline{F}(t)) - g(0_i, \underline{F}(t))$. For reliable systems $F_i(t)$ varies typically between $10^{-8}$ to $10^{-1}$ (where t can be thought of as mission time). Thus, subtracting a given increment $\Delta x$ from each basic event failure probability is not a good test for system upgrade because of the smallness and variability of $F_i(t)$. Instead, we must make fractional or relative changes in $F_i(t)$. This can be done by making $\Delta x$ a function of $F_i(t)$:

$$\Delta x = \gamma F_i(t),$$

where $\gamma$ is any given constant between 0 and 1. The expression

$$g[F_i(t), \underline{F}(t)] - g[F_i(t) - \gamma F_i(t), \underline{F}(t)]$$

identifies the event i that has the greatest effect in reducing $g(\underline{F}(t))$ when $F_i(t)$ is multiplied by a given constant $1 - \gamma$. In taking the limit as a approaches 1 in the above expression, we identify the difference as a differential quantity. Dividing the above expression by $1 - \gamma$, multiplying by $F_i(t)/F_i(t)$ (unity) we can then take the limit as $\gamma \to 1^-$,

$$\lim_{\gamma \to 1^-} F_i(t) \frac{g\{F_1(t), \ldots, F_i(t), \ldots, F_n(t)\} - g\{F_1(t), \ldots, \gamma F_i(t), \ldots F_n(t)\}}{F_i(t)(1 - \gamma)}$$

and identify the above quantity as being

$$F_i(t) \frac{\partial g(F(t))}{\partial F_i(t)} .$$

Note that the above expression is a function of $F_i(t)$.

The quantity that is physically measurable has a failure rate of $\lambda_i(t)$ as opposed to a failure probability of $F_i(t)$. Hence it is more meaningful to upgrade a system according to the following expression:

$$\lambda_i(t) \frac{\partial g(\underline{\lambda}(t))}{\partial \lambda_i(t)} .$$

It can be shown that when a first order expansion of $F_i(t) = 1 - \exp[-\int_0^t \lambda_i(t')dt']$ is a good approximation for $F_i(t)$, the above expression can be approximated by

$$F_i(t) \frac{\partial g(\underline{F}(t))}{\partial F_i(t)} .$$

Furthermore, this expression can be approximated by the Vesely-Fussell definition of importance when the rare event approximation is valid [7].

If the analyst assumes that the failure rates are proportional (i.e., assumption of proportional hazards), changes in $\lambda_i(t)$ are equivalent to changes in $\alpha_i$. Fractional or relative changes in $\alpha_i$ changes $g(\underline{\alpha}, q(t))$ incrementally at a rate

$$\alpha_i \frac{\partial g(\underline{\alpha}, q(t))}{\partial \alpha_i}$$

or fractionally at a rate

$$\frac{\alpha_i}{g(\underline{\alpha}, q(t))} \cdot \frac{\partial g(\underline{\alpha}, q(t))}{\partial \alpha_i} .$$

The last two expressions give the same relative ranking. The advantage of using the latter expression is that it yields numbers much closer to unity. It shall be denoted as the <u>upgrading function</u>.

If we identify a component failure with hazard rate $\alpha_i^I$ as the event for which

$$\frac{\alpha_i}{g(\underline{\alpha}, q(t))} \cdot \frac{\partial g(\underline{\alpha}, q(t))}{\partial \alpha_i}$$

is maximum, we may wish to replace the component with a more reliable component with a hazard rate of $\alpha_i^F$. If

$$\frac{\alpha_i}{g(\underline{\alpha}, q(t))} \cdot \frac{\partial g(\underline{\alpha}, q(t))}{\partial \alpha_i}$$

remains the maximum for all $\alpha_i$ between $\alpha_i^F$ and $\alpha_i^I$, then the optimal course of system upgrade has been chosen. However, if there is a value of $\alpha_i$, $\alpha_i^F \leq \alpha_i < \alpha_i^I$, in which another event j has a greater value:

$$\frac{\alpha_j}{g(\underline{\alpha}, q(t))} \cdot \frac{\partial g(\underline{\alpha}, q(t))}{\partial \alpha_j} > \frac{\alpha_i}{g(\underline{\alpha}, q(t))} \cdot \frac{\partial g(\underline{\alpha}, q(t))}{\partial \alpha_i} ,$$

then the absolute value of

$$g(\alpha_i \ldots \alpha_i^I, \ldots, \alpha_n, q(t)) - g(\alpha_i \ldots \alpha_i^F, \ldots, \alpha_n, q(t)) \text{ vs}$$

$$g(\alpha_i \ldots \alpha_j^I, \ldots, \alpha_n, q(t)) - g(\alpha_i \ldots \alpha_j^F, \ldots, \alpha_n, q(t)) \text{ must}$$

be calculated to determine the optimal choice of system upgrade. Cost constraints and other factors may limit the optimal course of system upgrade. Reference [7] gives an example of how the upgrading function can be applied to designing a system for safety.

6. Checklists for system diagnosis. The concept of importance can also be applied to after-the-fact investigation. If a fault tree can accurately simulate system failure and the failure rates of the basic events are known, then the fault tree can be quantitatively evaluated to determine the critical events. In the event of system/subsystem breakdown, a repair checklist can be generated for an operator to follow. The basic events on the checklist can be ordered according to their importance when system failure occurs.

All measures of importance mentioned in this paper, except Barlow and Proschan's, are independent of the order in which the basic events occur. It is felt that B-P's time series approach is the best method for identifying the most likely sequences of events leading to system failure.

An event that is a single-order cut set can be critical to system failure only if it causes the system to fail. Events contained in minimal cut sets, of order two or higher, can be critical to system failure in two ways. They can either cause the system to fail or fail prior to system failure and be critical when system failure occurs. Single-order cut sets are then ranked according to expression (5). Events contained in cut sets of order two or higher can be ranked according to expression (5) or (6).

Expressions (5) and (6) give additional information that cannot be found in other measures of importance, information such as the most efficient way of diagnosing system failure. For example, a component contained in a cut set of order two may have a relatively high probability of causing the system to fail. In turn, the failure of this component may be difficult to check. The operator can have the option of checking the other components that are apt to be critical when this component causes the system to fail and determining indirectly whether this component has failed.

The particular expression that is used to rank events on a checklist depends on when system failure is detected. If system failure can be observed only at the end of some time interval (O, T) then basic events can be critical to system failure at any time t in (O, T). In this case the basic events should be ranked according to expression (5) or (6). On the other hand, if system failure is observed at some instant of time, then an event must have caused the system to fail precisely at that time. These events should be ranked according to expressions analogous to expression (11). An example of the former case is a passive standby safety system that is checked at the end of the testing interval (O, T).

An example of the latter case is a continuously operating system that fails during operation.

The order in which the components are listed on the checklist should reflect the knowledge the operator gained about the system as he examined each component in the checklist. The ranking of the basic events should be done on a conditional basis. For example, if the operator finds that the first event has not occurred on the checklist, then the second event on the checklist should be the most critical to system failure, given that the first event has not occurred. In general, the $i^{th}$ event is most critical to system failure given that the first $i - 1$ events have not occurred. In generating the checklist, false alarms should be considered; i.e., the reliability of the monitoring device that indicates system failure should be considered. Reference [7] applies the above checklist generation scheme to a low pressure injection system, LPIS, at a nuclear power plant. The LPIS is a standby safety system, part of the emergency core cooling system.

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